A SIMPLIFIED METHOD FOR EVALUATING THE PRODUCT CUMULANTS OF SOME DISTRIBUTIONS ARISING FROM A SEQUENCE OF OBSERVATIONS

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1. INTRODUCTION

THE distributions of the number of joins or combinations, like *AA*, *AB*, *AC*; *BA*, *BB*, *BC*; ... between successive observations of a sequence in which *A*, *B*, *C*, ... occur with the respective probabilities p, q, r, ... have been discussed by the writer in detail in a number of previous communications (1949, 1952, 1958). The cumulants of these distributions were calculated by using the theorem that the r-th factorial moment is $r!$ times the expectation for r of the characters considered in the distribution (1958). The evaluation of the cumulants by this method was found to be cumbersome and tedious on account of heavy algebra. This procedure was simplified to a considerable extent in a subsequent publication (1952). It was shown that the r-th cumulant for the distribution of any of the characters (say *AA* joins) is given by

$$\frac{K_r}{r!} = \sum_{t=1}^{r} \frac{(n-t)\, K'_{s_1, s_2, \ldots, s_t}}{s_1!\, s_2! \ldots s_t!} \tag{1}$$

where t takes all values from 1 to r and s's take values subject to the condition that

$$s_1 + s_2 + \ldots + s_t = r,$$

and $K'_{s_1, s_2, \ldots s_t}$ represent the joint product cumulant of order r for t joins occurring in succession for the partition $s_1, s_2, \ldots, s_t$ of r. Thus, taking $x_1, x_2, \ldots, x_{n-1}$ to represent joins or combinations between 1st and 2nd; 2nd and 3rd; 3rd and 4th, etc.,....observations of the sequence and the expectation of the x's to be a, K'_{121} and K'_{22} are given by

$$K'_{121} = E(x_1 - a)(x_2 - a)^2 (x_3 - a) - 2[E(x_1 - a)(x_2 - a)]^2$$
$$- \{E(x_1 - a)(x_3 - a)\}\{E(x_2 - a)^2\} \quad (2)$$

$$K'_{22} = E(x_1 - a)^2 (x_2 - a)^2 - E(x_1 - a)^2 E(x_2 - a)^2$$
$$- 2\{E(x_1 - a)(x_2 - a)\}^2. \quad (3)$$

It can be easily seen that expectations like

$$E(x_1 - a)(x_l - a) = 0, \quad (4)$$

when l is greater than 2.

It would be noted that (2) and (3) have been obtained by using the expression

$$\log\left[1 + \sum_{r=1}^{\infty} \sum_{\substack{s_1 s_2, s_3 = 0 \\ s_1 + s_2 + s_3 = r}}^{r} \frac{\mu'_{s_1 s_2 s_3} t_1^{s_1} t_2^{s_2} t_3^{s_3}}{s_1!\, s_2!\, s_3!}\right]$$
$$= \sum_{r=1}^{\infty} \sum_{\substack{s_1, s_2, s_3 = 0 \\ s_1 + s_2 + s_3 = r}}^{r} \frac{K_{s_1 s_2 s_3} t_1^{s_1} t_2^{s_2} t_3^{s_3}}{s_1!\, s_2!\, s_3!} \quad (5)$$

Defining

$$\frac{(n-3)\, K'_{s_1 s_2 s_3}}{s_1!\, s_2!\, s_3!}$$

as the cumulant expectation of order r for the partition s_1, s_2 and s_3 from four consecutive observations, it can be seen that the r-th cumulant is equal to $r! \times$ the sum of the cumulant expectations for all the partitions of r. The partitions will obviously have $1, 2, \ldots, r$ parts. If there are t parts in a partition, the cumulants expectation depends on t joins obtained from $(t + 1)$ consecutive observations. The object of this paper is to extend the above procedure for the evaluation of product cumulants of joint distributions.

2. Derivation of Procedure

Let X and Y represent the number of AA and AB joins in a sequence of n observations. Taking for simplicity, the product cumulant $K_{31}(XY)$ it can be seen that

$$\frac{K_{31}(XY)}{3!\,1!} = \left\{K(X - \overline{n-1}\, p^2)^3 (Y - \overline{n-1}\, pq)\right\} \frac{1}{3!\,1!} \quad (6)$$

Assuming $z_r = (x_r - p^2)$ and $w_r = (y_r - pq)$, where x_r and y_r stand for AA and AB joins or combination joins between r and $(r+1)$-th observation, (6) reduces to

$$
\begin{aligned}
\frac{1}{3!\,1!}&\left\{K\left(\sum_1^{n-1} z_r\right)^3\left(\sum_1^{n-1} w_r\right)\right\} \\
&= \frac{(n-1)}{3!}K'(z_1^3w_1) + \frac{(n-2)}{3!}\{K'(z_1^3w_2) + K'(w_1z_2^3)\} \\
&\quad + \frac{(n-2)}{2!\,1!\,1!}\{K'(z_1^2\omega_1z_2) + K'(z_1^2z_2w_2) \\
&\qquad + K'(z_1w_1z_2^2) + K'(z_1z_2^2w_2)\} \\
&\quad + \frac{(n-3)}{2!\,1!\,1!}\{K'(z_1^2z_2w_3) + K'(z_1z_2^2w_3) + K'(z_1^2w_2z_3) \\
&\qquad + K'(z_1w_2z_3^2) + K'(w_1z_2^2z_3) \\
&\qquad + K'(w_1z_2z_3^2)\} \\
&\quad + (n-3)\{K'(z_1w_1z_2z_3) + K'(z_1z_2w_2w_3) \\
&\qquad + K'(z_1z_2z_3w_3) \\
&\quad + (n-4)\{K'(z_1z_2z_3w_4) + K'(z_1w_2z_3z_4) \\
&\qquad + K'(z_1z_2w_3z_4) + K'(w_1z_2z_3z_4).
\end{aligned} \tag{7}
$$

The contributions from the other terms are zero. Examining the left-hand side of (7), we note that it is the sum of the product cumulant expectations of the various connected combinations of x's and y's arising from the partition of the numbers 3 and 1.

In general, it will be seen that $K_{rs}(XY)/r!\,s!$ is equal to the sum of the cumulant expectations of order (r, s) for the connected joins arising from the partitions of r and s. The number of parts involved in this partitions ranges from 1 to $r + s$.

3. Applications

The product cumulants K_{31} and K_{22} for the joint distribution of the number of AA and AB joins between successive values for a sequence of n observations have been calculated by the methods developed in Section 2. Tables I and II give the various combinations involved in the calculations and their cumulants K'.

The final expressions for $K_{31}(XY)$ and $K_{22}(XY)$ are given below the tables. The value of $K_{13}(XY)$ obtained by this method is also given below Table I.

TABLE I

Combinations and Their Cumulant Expectations

K_{31}

Sl. No.	Combination	Cumulant	(3!×1! (Expected Value of Cumulant of Combination shown in Col. 2)	(No. of Combinations)
1	2	3	4	5
1	$x_1^3y_1$	$\mu(x_1^3y_1) - 3\mu(x_1^2)\mu(x_1y_1)$	$-p^3q + 6p^5q - 6p^7q$	$n-1$
2	$x_1^3y_2$	$\mu(x_1^3y_2) - 3\mu(x_1^2)\mu(x_1y_2)$	$p^2q - p^3q - 6p^4q + 6p^5q + 6p^6q - 6p^7q$	$n-2$
3	$y_1x_2^3$	$\mu(y_1x_2^3) - 3\mu(x_2^2)\mu(y_1x_2)$	$-p^3q + 6p^5q - 6p^7q$	$n-2$
4	$x_1^2y_1x_2$	$\mu(x_1^2y_1x_2) - \mu(x_1^2)\mu(y_1x_2) - 2\mu(x_1x_2)\mu(x_1y_1)$	$3(-p^4q + 2p^5q + 4p^6q - 6p^7q)$	$(n-2)$
5	$x_1^2x_2y_2$	$\mu(x_1^2x_2y_2) - \mu(x_1^2)\mu(x_2y_2) - 2\mu(x_1x_2)\mu(x_1y_2)$	$3(-2p^4q + 8p^6q - 6p^7q)$	$(n-2)$
6	$x_1y_1x_2^2$	$\mu(x_1y_1x_2^2) - \mu(x_2^2)\mu(x_1y_1) - 2\mu(x_1x_2)\mu(y_1x_2)$	$3(-p^4q + 2p^5q + 4p^6q - 6p^7q)$	$(n-2)$
7	$x_1x_2^2y_2$	$\mu(x_1x_2^2y_2) - \mu(x_2^2)\mu(x_1y_2) - 2\mu(x_1x_2)\mu(x_2y_2)$	$3(-2p^4q + 2p^5q + 6p^6q - 6p^7q)$	$(n-2)$
8	$x_1^2x_2y_3$	$\mu(x_1^2x_2y_3) - \mu(x_1^2)\mu(x_2y_3) - 2\mu(x_1x_2)\mu^*(x_1y_3)$	$3(p^3q - 2p^4q - p^5q + 4p^6q - 2p^7q)$	$(n-3)$
9	$x_1x_2^2y_3$	$\mu(x_1x_2^2y_3) - \mu(x_2^2)\mu^*(x_1y_3) - 2\mu(x_1x_2)\mu(x_2y_3)$	$3(p^3q - 2p^4q - 3p^5q + 8p^6q - 4p^7q)$	$(n-3)$
10	$x_1x_2x_3y_4$	$\mu(x_1x_2x_3y_4) - \mu(x_1x_2)\mu(x_3y_4) - \mu^*(x_1x_3)\mu(x_2y_4) - \mu^*(x_1y_4) \times \mu(x_2x_3)$	$6p^4q(1-p)^3$	$(n-4)$

11	$y_1x_2x_3x_4$	$\mu(y_1x_2x_3x_4)-\mu(y_1x_2)\,\mu(x_3x_4)-\mu^*(y_1x_3)\,\mu(x_2x_4)-\mu^*(y_1x_4)\times\mu(x_2x_3)$	$-6p^5q\,(1-p)^2$	$(n-4)$
12	$x_1y_2x_3x_4$	$\mu(x_1y_2x_3x_4)-\mu(x_1y_2)\,\mu(x_3x_4)-\mu^*(x_1x_3)\,\mu(y_2x_4)-\mu^*(x_1x_4)\times\mu(y_2x_3)$	$-6p^5q\,(1-p)^2$	$(n-4)$
13	$x_1x_2y_3x_4$	$\mu(x_1x_2y_3x_4)-\mu(x_1x_2)\,\mu(y_3x_4)-\mu^*(x_1y_3)\,\mu(x_2x_4)-\mu^*(x_1x_4)\times\mu(x_2y_3)$	$6\,(-p^5q+2p^6q-p^7q)$	$(n-4)$
14	$x_1^2y_2x_3$	$\mu(x_1^2y_2x_3)-\mu(x_1^2)\,\mu(y_2x_3)-2\mu^*(x_1x_3)\,\mu(x_1y_2)$	$3\,(-p^4q+p^5q+2p^6q-2p^7q)$	$(n-3)$
15	$x_1y_2x_3^2$	$\mu(x_1y_2x_3^2)-\mu(x_3^2)\,\mu(x_1y_2)-2\mu^*(x_1x_3)\,\mu(y_2x_3)$	$3\,\{-p^4q(1-p)\,(1-p^2)+p^6q\,(1-p)\}$	$(n-3)$
16	$y_1x_2^2x_3$	$\mu(y_1x_2^2x_3)-\mu(x_2^2)\,\mu^*(y_1x_3)-2\mu(x_2x_3)\,\mu(y_1x_2)$	$3\,(-p^4q+p^5q+4p^6q-4p^7q)$	$(n-3)$
17	$y_1x_2x_3^2$	$\mu(y_1x_2x_3^2)-\mu(x_3^2)\,\mu(y_1x_2)-2\mu(x_2x_3)\,\mu^*(y_1x_3)$	$3p^4q(-1+p+2p^2-2p^3)$	$(n-3)$
18	$x_1y_1x_2x_3$	$\mu(x_1y_1x_2x_3)-\mu(x_1y_1)\,\mu(x_2x_3)-\mu(x_1x_2)\mu^*(y_1x_3)-\mu^*(x_1x_3)\times\mu(y_1x_2)$	$6\,(-p^5q+3p^6q-2p^7q)$	$(n-3)$
19	$x_1x_2y_2x_3$	$\mu(x_1x_2y_2x_3)-\mu(x_1x_2)\,\mu(y_2x_3)-\mu(x_1y_2)\,\mu(x_2x_3)-\mu^*(x_1x_3)\times\mu(x_2y_2)$	$6\,(-2p^5q+6p^6q-4p^7q)$	$(n-3)$
20	$x_1x_2x_3y_3$	$\mu(x_1x_2x_3y_3)-\mu(x_1x_2)\,\mu(x_3y_3)-\mu^*(x_1x_3)\,\mu(x_2y_3)-\mu^*(x_1y_3)\times\mu(x_2x_3)$	$6\,(-2p^5q+4p^6q-2p^7q)$	$(n-3)$

$K_{31}(XY)$ = The sum of products of columns 4 and 5.
$= (n-2)\,p^2q+(3n-13)\,p^3q-2\,(21n-48)\,p^4q-6\,(5n-28)\,p^5q+18\,(15n-44)\,p^6q-6\,(35n-93)\,p^7q.$

$K_{13}(XY)$ calculated as above.
$= (n-2)\,p^2q-(3n-5)\,p^3q-12\,(2n-5)\,p^3q^2+12\,(5n-11)\,p^4q^2+6\,(15n-44)\,p^4q^3-6\,(35n-93)\,p^5q^3.$

* The product moments in these cases are zero.

TABLE II

Combinations and Their Cumulant Expectations

K_{22}

Sl. No.	Combination	Cumulant	(2!×2!) (Expected Value of Cumulant of Combination shown in Col. 2)	No. of Combination
1	2	3	4	5
1	$x_1^2y_1^2$	$\mu(x_1^2y_1^2)-\mu(x_1^2)\mu(y_1^2)-2[\mu(x_1y_1)]^2$	$-p^3q+2p^5q+2p^4q^2-6p^6q^2$	$n-1$
2	$x_1^2y_2^2$	$\mu(x_1^2y_2^2)-\mu(x_1^2)\mu(y_2^2)-2[\mu(x_1y_2)]^2$	$p^2q-p^3q-2p^3q^2-2p^4q+2p^5q+8p^5q^2-6p^6q^2$	$n-2$
3	$y_1^2x_2^2$	$\mu(y_1^2x_2^2)-\mu(y_1^2)\mu(x_2^2)-2[\mu(y_1x_2)]^2$	$-p^3q+2p^4q^2+2p^5q-6p^6q^2$	$n-2$
4	$x_1y_1^2x_2$	$\mu(x_1y_1^2x_2)-\mu(y_1^2)\mu(x_1x_2)-2\mu(x_1y_1)\mu(y_1x_2)$	$2(-p^4q+2p^5q+2p^5q^2-6p^6q^2)$	$(n-2)$
5	$x_1x_2y_2^2$	$\mu(x_1x_2y_2^2)-\mu(y_2^2)\mu(x_1x_2)-2\mu(x_1y_2)\mu(x_2y_2)$	$2(-2p^4q+2p^5q+6p^5q^2-6p^6q^2)$	$(n-2)$
6	$x_1^2y_1y_2$	$\mu(x_1^2y_1y_2)-\mu(x_1^2)\mu(y_1y_2)-2\mu(x_1y_1)\mu(x_1y_2)$	$2(-p^3q^2+2p^4q^2+2p^5q^2-6p^6q^2)$	$(n-2)$
7	$y_1x_2^2y_2$	$\mu(y_1x_2^2y_2)-\mu(x_2^2)\mu(y_1y_2)-2\mu(x_2y_2)\mu(y_1x_2)$	$2(2p^4q^2+2p^5q^2-6p^6q^2)$	$(n-2)$
8	$y_1^2x_2x_3$	$\mu(y_1^2x_2x_3)-\mu(y_1^2)\mu(x_2x_3)-2\mu(y_1x_2)\mu^*(y_1x_3)$	$2(-p^4q+p^5q+2p^5q^2-2p^6q^2)$	$(n-3)$
9	$x_1y_2^2x_3$	$\mu(x_1y_2^2x_3)-\mu(y_2^2)\mu^*(x_1x_3)-2\mu(x_1y_2)\mu(y_2x_3)$	$2(-p^4q+p^5q+4p^5q^2-4p^6q^2)$	$(n-3)$
10	$x_1x_2y_3^2$	$\mu(x_1x_2y_3^2)-\mu(y_3^2)\mu(x_1x_2)-2\mu^*(x_1y_3)\mu(x_2y_3)$	$2(p^3q-2p^4q-2p^4q^2+p^5q+4p^5q^2-2p^6q^2)$	$(n-3)$
11	$x_1^2y_2y_3$	$\mu(x_1^2y_2y_3)-\mu(x_1^2)\mu(y_2y_3)-2\mu(x_1y_2)\mu^*(x_1y_3)$	$2(-p^3q^2+p^4q^2+2p^5q^2-2p^6q^2)$	$(n-3)$
12	$y_1x_2^2y_3$	$\mu(y_1x_2^2y_3)-\mu(x_2^2)\mu^*(y_1y_3)-2\mu(y_1x_2)\mu(x_2y_3)$	$2(-p^3q^2+p^4q^2+4p^5q^2-4p^6q^2)$	$(n-3)$
13	$y_1y_2x_3^2$	$\mu(y_1y_2x_3^2)-\mu(x_3^2)\mu(y_1y_2)-2\mu^*(y_1x_3)\mu(y_2x_3)$	$2(p^4q^2-2p^6q^2)$	$(n-3)$
14	$x_1y_1x_2y_2$	$\mu(x_1y_1x_2y_2)-\mu(x_1y_1)\mu(x_2y_2)-\mu(x_1x_2)\mu(y_1y_2)-\mu(x_1y_2)\mu(y_1x_2)$	$4(4p^5q^2-6p^6q^2)$	$(n-2)$

15	$x_1y_1x_2y_3$	$\mu(x_1y_1x_2y_3) - \mu(x_1y_1)\mu(x_2y_3) - \mu(x_1x_2)\mu^*(y_1y_3) - \mu^*(x_1y_3)\mu(y_1x_2)$	$4(-p^4q^2 + 3p^5q^2 - 2p^6q^2)$	$(n-3)$
16	$x_1x_2y_2y_3$	$\mu(x_1x_2y_2y_3) - \mu(x_1x_2)\mu(y_2y_3) - \mu(x_1y_2)\mu(x_2y_3) - -\mu^*(x_1y_3)\mu(x_2y_2)$	$4(-2p^4q^2 + 6p^5q^2 - 4p^6q^2)$	$(n-3)$
17	$x_1y_1y_2x_3$	$\mu(x_1y_1y_2x_3) - \mu(x_1y_1)\mu(y_2x_3) - \mu(x_1y_2)\mu^*(y_1x_3) - \mu^*(x_1x_3)\mu(y_1y_2)$	$4(p^5q^2 - 2p^6q^2)$	$(n-3)$
18	$x_1y_2x_3y_3$	$\mu(x_1y_2x_3y_3) - \mu(x_1y_2)\mu(x_3y_3) - \mu^*(x_1x_3)\mu(y_2y_3) - \mu^*(x_1y_3)\mu(y_2x_3)$	$4(2p^5q^2 - 2p^6q^2)$	$(n-3)$
19	$y_1x_2y_2x_3$	$\mu(y_1x_2y_2x_3) - \mu(y_1x_2)\mu(y_2x_3) - \mu(y_1y_2)\mu(x_2x_3) - \mu^*(y_1x_3)\mu(x_2y_2)$	$4(2p^5q^2 - 4p^6q^2)$	$(n-3)$
20	$y_1x_2x_3y_3$	$\mu(y_1x_2x_3y_3) - \mu(y_1x_2)\mu(x_3y_3) - \mu^*(y_1x_3)\mu(x_2y_3) - \mu^*(y_1y_3)\mu(x_2x_3)$	$4(2p^5q^2 - 2p^6q^2)$	$(n-3)$
21	$x_1x_2y_3y_4$	$\mu(x_1x_2y_3y_4) - \mu(x_1x_2)\mu(y_3y_4) - \mu^*(x_1y_3)\mu(x_2y_4) - \mu^*(x_1y_4)\mu(x_2y_3)$	$4(-p^4q^2 + 2p^5q^2 - p^6q^2)$	$(n-4)$
22	$x_1y_2x_3y_4$	$\mu(x_1y_2x_3y_4) - \mu(x_1y_2)\mu(x_3y_4) - \mu^*(x_1x_3)\mu(y_2y_4) - \mu^*(x_1y_4)\mu(y_2x_3)$	$4(-p^4q^2 + 2p^5q^2 - p^6q^2)$	$(n-4)$
23	$x_1y_2y_3x_4$	$\mu(x_1y_2y_3x_4) - \mu(x_1y_2)\mu(y_3x_4) - \mu^*(x_1y_3)\mu(y_2x_4) - \mu^*(x_1x_4)\mu(y_2y_3)$	$4(p^5q^2 - p^6q^2)$	$(n-4)$
24	$y_1x_2y_3x_4$	$\mu(y_1x_2y_3x_4) - \mu(y_1x_2)\mu(y_3x_4) - \mu^*(y_1y_3)\mu(x_2x_4) - \mu^*(y_1x_4)\mu(x_2y_3)$	$4(p^5q^2 - p^6q^2)$	$(n-4)$
25	$y_1x_2x_3y_4$	$\mu(y_1x_2x_3y_4) - \mu(y_1x_2)\mu(x_3y_4) - \mu^*(y_1x_3)\mu(x_2y_4) - \mu^*(y_1y_4)\mu(x_2x_3)$	$4(-p^4q^2 + 2p^5q^2 - p^6q^2)$	$(n-4)$
26	$y_1y_2x_3x_4$	$\mu(y_1y_2x_3x_4) - \mu(y_1y_2)\mu(x_3x_4) - \mu^*(y_1x_3)\mu(y_2x_4) - \mu^*(y_1x_4)\mu(y_2x_3)$	$4(p^5q^2 - p^6q^2)$	$(n-4)$

$$K_{22}(XY) = (n-2)p^2q - (n+1)p^3q - 4(2n-5)p^3q^2 - 8(2n-5)p^4q - 2(5n-28)p^4q^2 + 4(5n-11)p^5q + 12(15n-44)p^5q^2 - 6(35n-93)p^6q^2.$$

* The product moments in these cases are zero.

It may be worthwhile to mention here that the method developed here for joins between consecutive observations can be extended for joins arising from alternate and other observations as well. Here also, we will have to find the sum of the cumulant expectations for all possible partitions giving connected combinations.

4. Summary

It has been established in this paper that the r–s product cumulant for the joint distribution of the number x's and y's, *i.e.*, AA and AB joins between consecutive values of a sequence of n observations is equal to $(r! \times s!)$ (the sum of the cumulant expectation of the various connected configurations of order r and s obtained for the various partitions of r and s). The method has been illustrated at length by calculating the values of K_{31} and K_{22}. It has been pointed out that this method can be extended for calculating the product cumulants of distributions which include combinations between successive, alternate and other observations as well.

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